Université de Bourgogne
UFR Sciences et Techniques

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Introduction to TQFT
Midterm exam 16/03/2023

Exercice 1:

Let F be a $n+1$-dimensional TQFT over a field $\mathbb{K}$. Let S be a closed oriented d-dimensional manifold, where $d < n$. For Σ a closed oriented $n-d$-dimensional manifold, let

$$G(\Sigma) = F(\Sigma \times S)$$

and for (M, Σ, Σ') a $n-d+1$ dimensional cobordism, we define

$$G(M) = F(M \times S),$$

where $M \times S$ stands for the natural $n+1$-dimensional cobordism $\Sigma \times S \to \Sigma' \times S$.

Show that G is a $n-d+1$-dimensional TQFT.

Exercise 2:

Let F be a $n+1$-dimensional TQFT, and let Σ be a closed oriented n-dimensional manifold. We denote by ρ the representation of $\mathrm{Diff}^+(\Sigma)$ such that $\rho(f) = F(C_f)$, where C_f is the mapping cylinder.

Let M_f be the closed orientable $n+1$-dimensional manifold defined by

$$M_f = \Sigma \times [0,1] / \{(x,1) \sim (f(x),0)\}.$$

Show that $F(M_f) = \mathrm{Tr}(\rho(f))$.

(*Hint:* Use the same strategy as to show $F(\Sigma \times S^1) = \dim(F(\Sigma))$.)

Exercise 3:

We define the category $PCob^{1+1}$ of pointed $1+1$ cobordisms as the symmetric monoidal category whose objects are 1-dimensional manifolds Σ containing a (possibly empty) collection of points P and whose morphisms between (Σ, P) and (Σ', P') are equivalence classes of cobordisms $M : \Sigma \to \Sigma'$ containing a 1-dimensional submanifold γ with $\partial\gamma = P \cup P'$. The monoidal product is disjoint union, and the twist is defined as usual. Let

$$F : PCob^{1+1} \to Vect_{\mathbb{K}}$$

be a symmetric monoidal functor.

Let $A = F((S^1, \emptyset))$ and $M = F((S^1, \{*\}))$, where $*$ is a point on S^1.

Explain why A has a natural commutative Frobenius algebra structure and define a natural A-module structure on M.

Exercise 4:

In this exercise, we will study the invariant of surfaces associated to a $1+1$-TQFT. Let $(A, \mu, 1, \delta, \varepsilon)$ be a commutative Frobenius algebra over a field $\mathbb{K}$, and let F_A be the associated $1+1$-TQFT. We write $f(g) = F_A(\Sigma_g)$ for the invariant of a closed connected orientable surface of genus g. We recall that $\Delta\mu : A \to A$ is called the handle operator, and that there is an element $w \in A$ such that $\Delta\mu$ is the multiplication by w.

(1) Show that the invariant of a closed orientable surface of genus g is $f(g) = F_A(\Sigma_g) = \varepsilon(w^g)$.

(2) Let n the dimension of A over $\mathbb{K}$. Show that $f(1) = n$. Show that there exists an integer $1 \leq d \leq n$ and coefficients $a_0, a_1, \ldots, a_{d-1} \in \mathbb{K}$ such that

$$f(m+d) = a_0 f(m) + a_1 f(m+1) + \ldots + a_{d-1} f(m+d-1),$$

for all integer $m \geq 0$. (*Hint:* Notice that the powers $1, w, \ldots, w^k, \ldots$ are linearly dependent)

(3) Let $A = \mathbb{K}$. Show that $\varepsilon \in A^*$ is a Frobenius form if and only if $\varepsilon(1) \neq 0$.

Show that $f(g) = \alpha^{1-g}$ for any $g \geq 0$, where $\alpha = \varepsilon(1)$. If $\mathbb{K} = \mathbb{C}$ show that the invariant F_A distinguishes all connected closed orientable surfaces if and only if α is not a root of unity. Does it distinguish all closed orientable surfaces ?

(4) Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}^*$.

Let (A, ε) be the Frobenius algebra such that $A = \mathbb{C}^n$ and $\varepsilon(t_1, \ldots, t_n) = \alpha_1 t_1 + \ldots + \alpha_n t_n$. Compute $F_A(\Sigma_g)$ for any $g \geq 0$.

(5) We say that $\alpha_1, \alpha_2 \in \mathbb{C}^*$ are algebraically independent if there is no non-zero Laurent polynomial $P \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$ such that $P(\alpha_1, \alpha_2) = 0$. Show that if $n = 2$ and α_1, α_2 are algebraically independent, then F_A distinguishes all closed orientable surfaces.